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PSEUDO-HOLOMORPHIC CURVES OF CONSTANT CURVATURE IN COMPLEX GRASSMANNIANS*

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ABSTRACT

In this paper we consider pseudo-holomorphic curves in complex Grassmiannians. Let $\varphi_0, \varphi_1, \cdots, \varphi_{\alpha_0} : S^2 \to G_{k,n}$ be a linearly full non-degenerate pseudo-holomorphic harmonic sequence, and let $\deg\varphi_\alpha$ and K_α be the degree and the Gauss curvature of φ_α ($\alpha = 0, 1, \cdots, \alpha_0$) respectively. Assume that $\varphi_0, \varphi_1, \cdots, \varphi_{\alpha_0}$ is totally unramified. Then we prove that (i) $\deg\varphi_\alpha = k(\alpha_0 - 2\alpha)$ for all $\alpha = 0, 1, \cdots, \alpha_0$; (ii) $K_\alpha = \frac{4}{k(\alpha_0 + 2\alpha(\alpha_0 - \alpha))}$ if K_α is constant for some $\alpha = 0, 1, \cdots, \alpha_0,$. We also give some conditions for pseudo-holomorphic curves with constant Kähler angle in complex Grassmiannians to be of constant curvature.

1. Introduction

Minimal surfaces of constant curvature in S^n have been classified completely (cf., [3]). Minimal 2-spheres of constant curvature in a complex projective space were also classified in [1] and [2], and rigidity theorems of holomorphic curves and conformal minimal spheres in a complex projective space were obtained (cf., [2], [5]). Our interest is to investigate conformal minimal spheres in a complex Grassmann manifold.

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In this paper we will use theory of harmonic maps to study geometry of pseudo-holomorphic curves in a complex Grassmann manifold. Many results of harmonic maps of surfaces into Lie groups and complex Grassmann manifolds were obtained (cf., [4], [7], [9], [13] and [14]). These results are used to study geometry of minimal spheres immersed in a complex Grassmann manifold. Harmonic sequence is one of the main tools we use in this paper.

Let φ be an immersion of a Riemann surface M into a Kähler manifold N. Its Kähler angle is defined to be the angle between $Jd\varphi(\partial/\partial x)$ and $d\varphi(\partial/\partial y)$, where $z = x + \sqrt{-1}y$ is a local complex coordinate on M and J denotes the complex structure of N. Chern and Wolfson (cf., [6]) pointed out the importance of the Kähler angle in the theory of minimal surfaces in Kähler manifolds. In 1988 J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward (cf., [2]) used the Kaehler angle to study conformal minimal spheres in complex projective spaces.

In [8] and [12] pseudo-holomorphic curves in a complex Grassmann manifold were studied. Recently, the classification of holomorphic spheres of constant curvature in $G_{2,5}$ was investigated (cf., [11]). In this paper we will discuss curvature, Kähler angle and degree of pseudo-holomorphic curves in a complex Grassmann manifold.

In the fourth section we will discuss the linearly full non-degenerate pseudo-harmonic sequence $\varphi_0, \varphi_1, \ldots, \varphi_{\alpha_0} : S^2 \to G_{k,n}$ (cf., [12]). Let $\deg\varphi_\alpha$ and K_α be the degree and the curvature of φ_α respectively. Suppose that $\varphi_0, \varphi_1, \ldots, \varphi_{\alpha_0}$ is a totally unramified harmonic sequence. We will prove that (i) $\deg\varphi_\alpha = k(\alpha_0 - 2\alpha)$ for every $\alpha = 0, 1, \ldots, \alpha_0$; (ii) $K_\alpha = \frac{4}{k(\alpha_0+2\alpha(\alpha_0-\alpha))}$ if K_α is constant for some $\alpha = 0, 1, \ldots, \alpha_0$ (see Theorem 4.1). By this result we get that the curvature of non-degenerate holomorphic curves of constant curvature in $G_{n,2n}$ equals $4/n$ (see Corollary 4.2). We will also give an example (the case of $G_{2,6}$) to show that it is possible that some elements in the non-degenerate harmonic sequence determined by a holomorphic curve of constant curvature in $G_{k,n}$ $(2 \le k \le n-2)$ are not of constant curvature.

In the fifth section we will discuss pseudo-holomorphic curves with constant Kähler angle in a complex Grassmann manifold and obtain Theorems 5.1 and 5.2. These theorems are generalizations of Bolton's results for complex projective spaces (cf., [2]).

2. Minimal immersions and harmonic sequences

Let $U(n)$ be the unitary group. Let M be a simply connected domain in the unit sphere S^2 and let $(z, \overline{z})$ a complex coordinate on M. We take the metric $ds_M^2 = dzd\overline{z}$ on M. Denote

$$\partial = \frac{\partial}{\partial z}, \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}}, \quad A_z = \frac{1}{2}s^{-1}\partial s, \quad A_{\overline{z}} = \frac{1}{2}s^{-1}\overline{\partial} s.$$

Let $s : M \to U(n)$ be a smooth map. Then s is a harmonic map if and only if it satisfies the following equation (cf., [13]):

$$\overline{\partial} A_z = [A_z, A_{\overline{z}}]. \tag{1}$$

If $s : S^2 \to U(n)$ is harmonic, then s is conformal and minimal. Let $\omega = g^{-1}dg$ be the Maurer–Cartan form on $U(n)$, and let $ds^2_{U(n)} = \frac{1}{8}\mathrm{tr}\omega\omega^*$ be the metric on $U(n)$. Then the metric induced by s on S^2 is locally given by

$$ds^2 = -\mathrm{tr}A_z A_{\overline{z}} dzd\overline{z}. \tag{2}$$

Let $G_{k,n}$ be the complex Grassmann manifold consisting of all complex k-dimensional subspaces in $\mathbf{C}^n$. Here we consider $G_{k,n}$ as the set of Hermitian orthogonal projections from $\mathbf{C}^n$ onto a k-dimensional subspace in $\mathbf{C}^n$, i.e., $G_{k,n} = \{\varphi$ is a Hermitian orthogonal projection onto a k-dimensional subspace in $\mathbf{C}^n\}$. Then $\varphi : S^2 \to G_{k,n}$ is a Hermitian orthogonal projection onto a k-dimensional subbundle $\eta \subset S^2 \times \mathbf{C}^n$, and $s = \varphi - \varphi^\perp$ is a map from S^2 into $U(n)$. It is well-known that φ is harmonic if and only if s is harmonic. φ is a **holomorphic curve** (or an **anti-holomorphic curve**) in $G_{k,n}$ if and only if $\varphi^\perp \overline{\partial}\varphi = 0$ (or $\varphi^\perp \partial\varphi = 0$).

By using φ, a harmonic sequence (cf., [7], [14]) is derived as follows:

$$\varphi = \varphi_0 \stackrel{\partial'}{\longmapsto} \varphi_1 \stackrel{\partial'}{\longmapsto} \cdots \stackrel{\partial'}{\longmapsto} \varphi_\alpha \stackrel{\partial'}{\longmapsto} \cdots, \tag{3}$$

$$\varphi = \varphi_0 \stackrel{\partial''}{\longmapsto} \varphi_{-1} \stackrel{\partial''}{\longmapsto} \cdots \stackrel{\partial''}{\longmapsto} \varphi_{-\alpha} \stackrel{\partial''}{\longmapsto} \cdots, \tag{4}$$

where $\varphi_\alpha = \partial'\varphi_{\alpha-1}$ and $\varphi_{-\alpha} = \partial''\varphi_{-\alpha+1}$ are Hermitian orthogonal projections from $S^2 \times \mathbf{C}^n$ onto $\underline{\mathrm{Im}}\left(\varphi^\perp_{\alpha-1}\partial\varphi_{\alpha-1}\right)$ and $\underline{\mathrm{Im}}\left(\varphi^\perp_{-\alpha+1}\overline{\partial}\varphi_{-\alpha+1}\right)$ respectively, $\alpha = 1, 2, \ldots$.

Let $k_\alpha = \mathrm{rank}\,(\varphi_\alpha)$ and $k_{-\alpha} = \mathrm{rank}\,(\varphi_{-\alpha})$ for $\alpha = 0, 1, \ldots$. We say that φ_α in (3) (resp., $\varphi_{-\alpha}$ in (4)) is **non-degenerate** if $k_\alpha = k_{\alpha+1}$ (resp., $k_{-\alpha} = k_{-\alpha-1}$). If φ_0 is a holomorphic curve in (3), then there are finite elements, which are mutually orthogonal, in (3).

Suppose that $\varphi : S^2 \to G_{k,n}$ is a pseudo-holomorphic curve (φ is obtained by some holomorphic curves φ_0 via ∂'), i.e. φ belongs to the following harmonic sequence (cf., [8])

$$0 \xmapsto{\partial'} \varphi_0 \xmapsto{\partial'} \varphi_1 \xmapsto{\partial'} \cdots \xmapsto{\partial'} \varphi = \varphi_\alpha \xmapsto{\partial'} \cdots \xmapsto{\partial'} \varphi_{\alpha_0} \xmapsto{\partial'} 0. \tag{5}$$

If $k_\alpha = k_\beta$ for all $\alpha, \beta = 0, \ldots, \alpha_0$ in (5), then φ is also obtained by the anti-holomorphic curve φ_{α_0} via ∂'', and (5) is uniquely determined by φ (cf., [7]). We say that (5) is a **non-degenerate harmonic sequence associate to** φ and α_0 is the **length** of this non-degenerate harmonic sequence.

Definition: Let $\varphi : S^2 \to G_{k,n}$ be a map. φ is **linearly full** if $\underline{\mathrm{Im}}(\varphi)$ cannot be contained in any proper trivial subbundle $S^2 \times \mathbf{C}^m$ of $S^2 \times \mathbf{C}^n$ $(m < n)$.

In this paper, we always assume that φ is a linearly full pseudo-holomorphic curve. Under this assumption, we have $k_0 + \cdots + k_{\alpha_0} = n$ in (5).

For the harmonic sequence (5) we choose the local unitary frame $e_1, e_2, \ldots, e_n$ of $S^2 \times \mathbf{C}^n$ such that $e_{k_0+\cdots+k_{\alpha-1}+1}, \ldots, e_{k_0+\cdots+k_\alpha}$ locally span subbundle $\underline{\mathrm{Im}}\left(\varphi_{\alpha-1}^{\perp}\partial\varphi_{\alpha-1}\right)$ of $S^2 \times \mathbf{C}^n$, where $\alpha = 1, 2, \ldots, \alpha_0$.

Let $W_\alpha = \left(e_{k_0+\ldots+k_{\alpha-1}+1}, \ \ldots, \ e_{k_0+\ldots+k_\alpha}\right)$ be an $(n \times k_\alpha)$-matrix for $\alpha = 1, \ldots, \alpha_0$, and let $W_0 = (e_1, \ \ldots, \ e_{k_0})$ be an $(n \times k_0)$-matrix. Then, we have

$$\varphi_\alpha = W_\alpha W_\alpha^*, \tag{6}$$

$$W_\alpha^* W_\alpha = I_{k_\alpha \times k_\alpha}, \quad W_\alpha^* W_{\alpha+1} = 0, \quad W_\alpha^* W_{\alpha-1} = 0. \tag{7}$$

By (7), a straightforward computation shows that

$$\begin{cases} \partial W_\alpha = W_{\alpha+1}\Omega_\alpha + W_\alpha \Psi_\alpha, \\ \overline{\partial} W_\alpha = -W_{\alpha-1}\Omega_{\alpha-1}^* - W_\alpha \Psi_\alpha^*, \end{cases} \tag{8}$$

where Ω_α is a $(k_{\alpha+1} \times k_\alpha)$-matrix and Ψ_α is a $(k_\alpha \times k_\alpha)$-matrix for $\alpha = 0, 1, 2, \ldots, \alpha_0$, and $\Omega_{-1} = \Omega_{\alpha_0} = 0$.

It is very evident that integrability conditions for (8) are

$$\overline{\partial}\Omega_\alpha = \Psi_{\alpha+1}^*\Omega_\alpha - \Omega_\alpha \Psi_\alpha^*, \tag{9}$$

$$\overline{\partial}\Psi_\alpha + \partial\Psi_\alpha^* = \Omega_\alpha^*\Omega_\alpha + \Psi_\alpha^*\Psi_\alpha - \Omega_{\alpha-1}\Omega_{\alpha-1}^* - \Psi_\alpha\Psi_\alpha^*. \tag{10}$$

By (8), $A_z^{(\alpha)}$ and $A_{\overline{z}}^{(\alpha)}$ for φ_α are given by

$$A_z^{(\alpha)} = -W_\alpha \Omega_{\alpha-1} W_{\alpha-1}^* - W_{\alpha+1}\Omega_\alpha W_\alpha^*, \tag{11}$$

$$A^{(\alpha)}_{\overline{z}} = W_\alpha \Omega^*_\alpha W^*_{\alpha+1} + W_{\alpha-1}\Omega^*_{\alpha-1}W^*_\alpha. \tag{12}$$

Now we assume that φ_α is non-degenerate, then $|\det\Omega_\alpha|^2 dz^{k_\alpha} d\overline{z}^{k_\alpha}$ is a well-defined invariant and has only isolated zeros on S^2.

Set $l_\alpha = \mathrm{tr}(\Omega_\alpha\Omega^*_\alpha)$. It can easily be verified that (cf., [12])

$$l_\alpha = \mathrm{tr}\left(\varphi^\perp_\alpha \partial\varphi_\alpha \overline{\partial}\varphi_\alpha\right), \quad l_{\alpha-1} + l_\alpha = -\mathrm{tr}\left(A^{(\alpha)}_z A^{(\alpha)}_{\overline{z}}\right),$$

and we have

$$2\partial\overline{\partial}\log|\det\Omega_\alpha| = l_{\alpha-1} - 2l_\alpha + l_{\alpha+1}. \tag{13}$$

Remark: If (5) is a non-degenerate harmonic sequence, then (13) holds for all $\alpha = 0, 1, \ldots, \alpha_0 - 1$, where $l_{-1} = l_{\alpha_0} = 0$. When $k_\alpha = 1$ for all α, then $l_\alpha = |\det\Omega_\alpha|^2$, and (13) is just the **unintegrated Plücker formulae** for l_α derived by Bolton, Jensen, Rigoli and Woodward (cf., [2]).

3. Kähler angle, curvature and degree

If $\varphi : M \to G_{k,n}$ is a conformal immersion of a Riemann surface M, we define the Kähler angle of φ to be the function $\theta : M \to [0, \pi]$ given in terms of a complex coordinate z on M by (cf., [2], [6])

$$\tan\frac{\theta(p)}{2} = \frac{|d\varphi(\partial/\partial\overline{z})|}{|d\varphi(\partial/\partial z)|}, \quad p \in M. \tag{14}$$

It is clear that θ is globally defined and is smooth at p unless $\theta(p) = 0$ or π. φ is holomorphic if and only if $\theta(p) = 0$ for all $p \in M$, while φ is anti-holomorphic if and only if $\theta(p) = \pi$ for all $p \in M$.

Let $\varphi : S^2 \to G_{k,n}$ be a conformal minimal immersion with the harmonic sequence (5). Then, each $\varphi_\alpha : S^2 \to G_{k_\alpha,n}$ in (5) is also a conformal minimal immersion. So there exists a finite set X_α (cf., [2]) such that the Kähler angle

$$\theta_\alpha : S^2 \setminus X_\alpha \to [0, \pi]$$

is well-defined, and is smooth on $S^2 \setminus X_\alpha$.

Set $t_\alpha = (\tan(\theta_\alpha/2))^2$. Then, in terms of a local complex coordinate z,

$$t_\alpha = l_{\alpha-1}/l_\alpha.$$

Let ds^2_α be the metric on $S^2 \setminus X_\alpha$ induced by φ_α. Then

$$ds^2_\alpha = (l_{\alpha-1} + l_\alpha)dzd\overline{z}.$$

The Laplacian $\triangle_\alpha$ and the curvature K_α of ds^2_α are given by

$$\triangle_\alpha = \frac{4}{l_{\alpha-1}+l_\alpha}\partial\overline{\partial}, \quad K_\alpha = -\frac{2}{l_{\alpha-1}+l_\alpha}\partial\overline{\partial}\log(l_{\alpha-1}+l_\alpha),$$

and the area form dv_α by

$$dv_\alpha = (l_{\alpha-1}+l_\alpha)\frac{d\overline{z}\wedge dz}{2\sqrt{-1}}.$$

Let $\varphi^{(\alpha)} = \varphi_0 \oplus \cdots \oplus \varphi_\alpha$. Then $\varphi^{(\alpha)}$ is holomorphic, and $\partial'\varphi^{(\alpha)} = \varphi_{\alpha+1}$ (cf., [12]).

Now choose $k_{(\alpha)}$ holomorphic $\mathbf{C}^n$-valued functions $f_1, \dots, f_{k_{(\alpha)}}$ so that they locally span $\underline{\mathrm{Im}}\left(\varphi^{(\alpha)}\right)$, where $k_{(\alpha)} = k_0 + \cdots + k_\alpha$. Let

$$F^{(\alpha)}\rho_\alpha(z) = f_1 \wedge \cdots \wedge f_{k_{(\alpha)}},$$

where $\rho_\alpha(z)$ is the greatest common divisor of the $\binom{n}{k_{(\alpha)}}$ components of $f_1 \wedge \cdots \wedge f_{k_{(\alpha)}}$. Then we have $|\rho_\alpha(z)|^2 \left|F^{(\alpha)}\right|^2 = \left|f_1 \wedge \cdots \wedge f_{k_{(\alpha)}}\right|^2$, and

$$F^{(\alpha)} : S^2 \to \mathbf{C}^{\binom{n}{k_{(\alpha)}}}$$

is a nowhere zero holomorphic curve.

Consider the composite of the Plücker embedding (cf., [10]) with $\varphi^{(\alpha)}$,

$$\left[F^{(\alpha)}\right] : S^2 \to CP^{\binom{n}{k_{(\alpha)}}-1}, \tag{15}$$

which is a holomorphic isometry. Then $\left[F^{(\alpha)}\right]^* ds^2_{CP^{\binom{n}{k_{(\alpha)}}-1}} = l_\alpha dz d\overline{z}$, and we have

$$\partial\overline{\partial}\log\left|F^{(\alpha)}\right|^2 = l_\alpha, \tag{16}$$

and the degree δ_α of $\left[F^{(\alpha)}\right]$ is given by

$$\delta_\alpha = \frac{1}{2\pi\sqrt{-1}}\int_{S^2}\partial\overline{\partial}\log\left|F^{(\alpha)}\right|^2 d\overline{z}\wedge dz = \frac{1}{2\pi\sqrt{-1}}\int_{S^2} l_\alpha d\overline{z}\wedge dz, \tag{17}$$

which is equal to the degree of the polynomial function $F^{(\alpha)}$ in z. Then we have

Proposition 3.1 ([12]): *Let $\varphi : S^2 \to G_{k,n}$ be a linearly full non-degenerate pseudo-holomorphic curve. Assume that φ is the α-th element of* (5) *for some $\alpha = 0, 1, \dots, \alpha_0 - 1$, then*

$$\frac{1}{2\pi\sqrt{-1}}\int_{S^2}\partial\overline{\partial}\log|\det\Omega_\alpha|^2 d\overline{z}\wedge dz = \delta_{\alpha-1} - 2\delta_\alpha + \delta_{\alpha+1}. \tag{18}$$

Remark: If (5) is a non-degenerate harmonic sequence, then (18) holds for all $\alpha = 0, 1, \dots, \alpha_0 - 1$, where $\delta_{-1} = \delta_{\alpha_0} = 0$; In particular, when $k_0 = \cdots = k_{\alpha_0} = 1$, (18) is the global Plücker formula (cf., [2], [10]). If $\varphi_\alpha : S^2 \to G_{k_\alpha,n}$ is a pseudo-holomorphic with constant Kähler angle, then $t_\alpha = \delta_{\alpha-1}/\delta_\alpha$.

We choose k_α local sections $h_1^\alpha, \dots, h_{k_\alpha}^\alpha$, which locally span $\underline{\mathrm{Im}}(\varphi_\alpha)$, such that

$$f_1 \wedge \cdots \wedge f_{k_{(\alpha)}} = f_1 \wedge \cdots \wedge f_{k_{(\alpha-1)}} \wedge h_1^\alpha \wedge \cdots \wedge h_{k_\alpha}^\alpha.$$

We set

$$|\varphi_\alpha|^2 = \left|h_1^\alpha \wedge \cdots \wedge h_{k_\alpha}^\alpha\right|^2. \tag{19}$$

Then $|\varphi_\alpha|^2$ has some isolated zeros on S^2. Evidently, we have

$$|\rho_\alpha(z)|^2 \left|F^{(\alpha)}\right|^2 = |\rho_{\alpha-1}(z)|^2 \left|F^{(\alpha-1)}\right|^2 |\varphi_\alpha|^2. \tag{20}$$

Specially, we have $|\varphi_0|^2 = |f_1 \wedge \cdots \wedge f_{k_0}|^2 = |\rho_0(z)|^2 \left|F^{(0)}\right|^2$. Hence

$$|\rho_\alpha(z)|^2 \left|F^{(\alpha)}\right|^2 = |\varphi_0|^2 \cdots |\varphi_\alpha|^2.$$

From (20) it follows that

$$\partial\overline{\partial} \log |\varphi_\alpha|^2 = l_\alpha - l_{\alpha-1}, \tag{21}$$

where $\alpha = 0, 1, \dots, \alpha_0$.

Then we have the following lemma.

LEMMA 3.2: *Let $\varphi = \varphi_\alpha : S^2 \to G_{k_\alpha,n}$ be a linearly full non-degenerate pseudo-holomorphic curve. Then we have*

$$|\det \Omega_\alpha|^2 = \frac{|\varphi_{\alpha+1}|^2}{|\varphi_\alpha|^2}. \tag{22}$$

Proof. From (8) it follows that

$$\Omega_\alpha = W_{\alpha+1}^* \partial W_\alpha$$

Let $H_\alpha = \left(h_1^\alpha, \dots, h_{k_\alpha}^\alpha\right)$ and $H_{\alpha+1} = \left(h_1^{\alpha+1}, \dots, h_{k_{\alpha+1}}^{\alpha+1}\right)$. Then there are invertible $(k_\alpha \times k_\alpha)$-matrix A_α and $(k_{\alpha+1} \times k_{\alpha+1})$-matrix $A_{\alpha+1}$ such that

$$H_\alpha = W_\alpha A_\alpha, \quad H_{\alpha+1} = W_{\alpha+1} A_{\alpha+1}.$$

Since φ_α is non-degenerate, then we have $\partial H_\alpha = H_{\alpha+1} + H_\alpha B_\alpha$, and

$$\Omega_\alpha = \left(A_{\alpha+1}^*\right)^{-1} H_{\alpha+1}^* \partial H_\alpha A_\alpha^{-1} = (A_{\alpha+1}^*)^{-1} H_{\alpha+1}^* H_{\alpha+1} A_\alpha^{-1}.$$

Hence we have

$$\det \Omega_\alpha = \frac{\det(A^*_{\alpha+1})^{-1}\det(H^*_{\alpha+1}H_{\alpha+1})}{\det A_\alpha} = \frac{\det A_{\alpha+1}}{\det A_\alpha}.$$

The result is immediate from

$$|\varphi_\alpha|^2 = \det\left(A^*_\alpha A_\alpha\right) \quad \text{and} \quad |\varphi_{\alpha+1}|^2 = \det\left(A^*_{\alpha+1}A_{\alpha+1}\right). \qquad \blacksquare$$

Let L_α be the line bundle spanned by $h^\alpha_1 \wedge \cdots \wedge h^\alpha_{k_\alpha}$. If $c_1(L_\alpha)$ denotes the first Chern class of the line bundle L_α, then for $\alpha = 0, 1, \ldots, \alpha_0$

$$c_1(L_\alpha) = \frac{-1}{2\pi\sqrt{-1}}\int_{S^2} \partial\overline{\partial}\log|\varphi_\alpha|^2 d\overline{z}\wedge dz.$$

Hence

$$c_1(L_\alpha) = \frac{1}{2\pi\sqrt{-1}}\int_{S^2}(l_{\alpha-1} - l_\alpha)d\overline{z}\wedge dz \tag{23}$$

and

$$c_1(L_{\alpha+1}) - c_1(L_\alpha) = -\frac{1}{2\pi\sqrt{-1}}\int_{S^2}(l_{\alpha-1} - 2l_\alpha + l_{\alpha+1})d\overline{z}\wedge dz. \tag{24}$$

Thus, it follows from (17), (23) and (24) that

$$c_1(L_\alpha) = \delta_{\alpha-1} - \delta_\alpha, \tag{25}$$

$$c_1(L_{\alpha+1}) - c_1(L_\alpha) = -(\delta_{\alpha-1} - 2\delta_\alpha + \delta_{\alpha+1}), \tag{26}$$

where $\alpha = 0, 1, \ldots, \alpha_0$, and $\delta_{-1} = \delta_{\alpha_0} = 0$.

Let M denote a compact Riemann surface and let $\varphi : M \to G_{k,n}$ be a smooth map.

Definition ([4]): The **degree** of φ, denoted $\deg\varphi$ is the degree of the induced map $\varphi^* : H^2(G_{k,n}, \mathbf{Z}) \cong \mathbf{Z} \to H^2(M, \mathbf{Z}) \cong \mathbf{Z}$ on second cohomology. Note that a holomorphic map has nonnegative degree (cf., [9]).

By Lemma 5.1 of [4] we have

Proposition 3.3: *Let $\varphi = \varphi_\alpha : S^2 \to G_{k,n}$ be a linearly full pseudo-holomorphic curve. Then $\deg\varphi = \delta_\alpha - \delta_{\alpha-1}$ and $\deg\varphi^\perp = -\deg\varphi$.*

By Proposition 3.1, Lemma 3.2 and Proposition 3.3 we get

Lemma 3.4: *Let $\varphi = \varphi_\alpha : S^2 \to G_{k_\alpha,n}$ be a linearly full non-degenerate pseudo-holomorphic curve. Suppose that $|\det\Omega_\alpha|^2 dz^{k_\alpha} d\overline{z}^{k_\alpha} \neq 0$, then*

(i) $\delta_{\alpha-1} - 2\delta_\alpha + \delta_{\alpha+1} = -2k_\alpha$;

(ii) $\deg\varphi_{\alpha+1} - \deg\varphi_\alpha = -2k_\alpha$.

Proof. By definition (8) of Ω_α it can be easily checked that $\Omega_\alpha dz$ is a matrix-valued differential form. Let $|\omega_j^\alpha|^2$ ($j = 1, \ldots, k_\alpha$) be eigenvalues of $\Omega_\alpha^*\Omega_\alpha$. Then $|\omega_j^\alpha|^2 dz d\overline{z}$ is globally defined on S^2. Since $|\det\Omega_\alpha|^2 dz^{k_\alpha} d\overline{z}^{k_\alpha} \neq 0$, we get for $j = 1, \ldots, k_\alpha$, $|\omega_j^\alpha|^2 dz d\overline{z} \neq 0$. Then by the Gauss–Bonnet–Chern theorem (cf.,[15]) we have

$$\frac{1}{2\pi\sqrt{-1}}\int_{S^2}\partial\overline{\partial}\log|\omega_j^\alpha|^2 d\overline{z}\wedge dz = -2.$$

Therefore, it follows that

$$\frac{1}{2\pi\sqrt{-1}}\int_{S^2}\partial\overline{\partial}\log|\det\Omega_\alpha|^2 d\overline{z}\wedge dz = \frac{1}{2\pi\sqrt{-1}}\sum_{j=1}^{k_\alpha}\int_{S^2}\partial\overline{\partial}\log|\omega_{jj}^\alpha|^2 d\overline{z}\wedge dz = -2k_\alpha.$$

By Proposition 3.1 we have

$$\delta_{\alpha-1} - 2\delta_\alpha + \delta_{\alpha+1} = -2k_\alpha.$$

By Lemma 3.2 and Proposition 3.3 we get

$$\deg\varphi_{\alpha+1} - \deg\varphi_\alpha = -2k_\alpha. \qquad \blacksquare$$

We remark that the energy $E(L_\alpha)$ of the map $\varphi_\alpha : S^2 \to G_{k,n}$ defined by

$$E(L_\alpha) = \frac{1}{2\pi\sqrt{-1}}\int_{S^2}(l_\alpha + l_{\alpha-1})d\overline{z}\wedge dz,$$

is also an integer, namely,

$$E(L_\alpha) = \delta_\alpha + \delta_{\alpha-1}. \tag{27}$$

4. Minimal spheres of constant curvature with non-degenerate associated harmonic sequence

Let $\varphi : S^2 \to G_{k,n}$ be a linearly full pseudo-holomorphic curve with non-degenerate associated harmonic sequence ($n = k(\alpha_0 + 1)$ in this case)

$$\varphi_0, \varphi_1, \ldots, \varphi_{\alpha_0} : S^2 \to G_{k,n}. \tag{28}$$

If $\varphi_0, \varphi_1, \ldots, \varphi_n : S^2 \to CP^n$ is the Veronese sequence (cf., [1], [2]), then for all $\alpha = 0, 1, \ldots, n$,

$$\left|F^{(\alpha)}\right|^2 = a_\alpha(1 + z\overline{z})^{(\alpha+1)(n-\alpha)}, \quad |\varphi_\alpha|^2 = b_\alpha(1 + z\overline{z})^{n-2\alpha},$$

and for all $\alpha = 0, 1, \ldots, n-1$,

$$|\det \Omega_\alpha|^2 dz d\overline{z} = c_\alpha (1 + z\overline{z})^{-2} dz d\overline{z} \neq 0,$$

where above $a_\alpha, b_\alpha, c_\alpha$ are constant.

The Veronese sequence is a totally unramified harmonic sequence by Bolton's definition in [2]. Similarly, for general $G_{k,n}$, we say that non-degenerate harmonic sequence (28) is a **totally unramified harmonic sequence** if $|\det \Omega_\alpha|^2 dz^k d\overline{z}^k \neq 0$ for all $\alpha = 0, 1, \ldots, \alpha_0 - 1$.

We say that (28) is a **harmonic sequence of constant curvature** if each map of (28) is of constant curvature.

In the following we prove that

THEOREM 4.1: *Let $\varphi : S^2 \to G_{k,n}$ be a linearly full pseudo-holomorphic curve with non-degenerate associated harmonic sequence* (28). Suppose that (28) is a totally unramified harmonic sequence, then

(i) $\deg \varphi_\alpha = k(\alpha_0 - 2\alpha)$ for all $\alpha = 0, 1, \ldots, \alpha_0$;
(ii) $K_\alpha = \frac{4}{k(\alpha_0 + 2\alpha(\alpha_0 - \alpha))}$ if K_α is constant for some $\alpha = 0, 1, \cdots, \alpha_0$.

Proof. By Lemma 3.4 we can get

$$\delta_\alpha = k(\alpha + 1)(\alpha_0 - \alpha),$$

for all $\alpha = 0, 1, \ldots, \alpha_0$.

By Proposition 3.3 we have

$$\deg \varphi_\alpha = \delta_\alpha - \delta_{\alpha-1} = k(\alpha_0 - 2\alpha).$$

If the Gaussian curvature K_α of φ_α is constant, then K_α is given by

$$K_\alpha = \frac{4}{\delta_{\alpha-1} + \delta_\alpha} = \frac{4}{k(\alpha_0 + 2\alpha(\alpha_0 - \alpha))}. \qquad \blacksquare$$

The following corollary is an immediate consequence of Theorem 4.1.

COROLLARY 4.2: *Let $\varphi : S^2 \to G_{n,2n}$ be a holomorphic curve of constant curvature. Suppose that φ is non-degenerate, then $K(\varphi) = 4/n$.*

Note: It can easily be checked that $|\det \Omega_\alpha|^2 dz^k d\overline{z}^k = (1 + z\overline{z})^{-2k} dz^k d\overline{z}^k$ for all $\alpha = 0, 1, \ldots, \alpha_0 - 1$ if (28) is a harmonic sequence of constant curvature.

It is well-known that for $k = 1$, (5) determined by a conformal minimal sphere of constant curvature in a complex projective space is a harmonic sequence of constant curvature. Furthermore, (5) is the Veronese sequence, up to an isometry of CP^n (cf.,[1], [2]). In the following, we will give an example to conclude that it is a possibility that this result doesn't hold for $G_{k,n}$ $(2 \leq k \leq n-2)$.

We consider a non-degenerate harmonic sequence in $G_{2,6}$ $(\alpha_0 = 2)$:

$$0 \xmapsto{\partial'} \varphi_0 \xmapsto{\partial'} \varphi_1 \xmapsto{\partial'} \varphi_2 \xmapsto{\partial'} 0,$$

where $\varphi_0 : S^2 \to G_{2,6}$ is a holomorphic curve of constant curvature.

We will construct φ_0 such that φ_1 and φ_2 are not maps of constant curvature.

Set $\underline{\mathrm{Im}}(\varphi_\alpha) = \mathrm{span}\{f_\alpha(z,\overline{z}), g_\alpha(z,\overline{z})\}$, where $f_\alpha(z,\overline{z})$ and $g_\alpha(z,\overline{z})$ are two (local) linearly independent sections of $S^2 \times \mathbf{C}^6$, $\alpha = 0, 1, 2$.

We choose two local sections $f_0(z)$ and $g_0(z)$ of $\underline{\mathrm{Im}}(\varphi_0)$, which is a holomorphic subbundle of $S^2 \times \mathbf{C}^6$, as follows:

$$f_0(z) = \left(1,\ 0,\ \frac{1}{\sqrt{2}}z,\ \frac{\sqrt{31}}{2\sqrt{7}}z^2,\ \frac{9}{2\sqrt{7}}z^2,\ 0\right), \quad g_0(z) = \left(0,\ 1,\ 0,\ 0,\ \frac{\sqrt{7}}{\sqrt{2}}z,\ \frac{1}{2}z^2\right).$$

From $\underline{\mathrm{Im}}(\varphi_1) = \underline{\mathrm{Im}}\left(\varphi_0^{\perp}\partial\varphi_0\right)$ it follows that two local sections $f_1(z,\overline{z})$ and $g_1(z,\overline{z})$ of $\underline{\mathrm{Im}}(\varphi_1)$ are given by

$$f_1(z,\overline{z}) = \left(x_1^{(1)},\ x_2^{(1)},\ x_3^{(1)},\ x_4^{(1)},\ x_5^{(1)},\ x_6^{(1)}\right),$$
$$g_1(z,\overline{z}) = \left(y_1^{(1)},\ y_2^{(1)},\ y_3^{(1)},\ y_4^{(1)},\ y_5^{(1)},\ y_6^{(1)}\right),$$

where

$$x_1^{(1)} = -\frac{\overline{z}\left(4 + 78z\overline{z} + 63z^2\overline{z}^2 + 16z^3\overline{z}^3\right)}{8}, \qquad x_2^{(1)} = -\frac{9z\overline{z}(4+z\overline{z})}{4\sqrt{2}},$$

$$x_3^{(1)} = \frac{8 + 28z\overline{z} - 30z^2\overline{z}^2 - 31z^3\overline{z}^3 - 8z^4\overline{z}^4}{8\sqrt{2}},$$

$$x_4^{(1)} = \frac{\sqrt{31}z\left(16 + 60z\overline{z} + 18z^2\overline{z}^2 + z^3\overline{z}^3\right)}{16\sqrt{7}},$$

$$x_5^{(1)} = \frac{9z\left(16 + 4z\overline{z} + 4z^2\overline{z}^2 + z^3\overline{z}^3\right)}{16\sqrt{7}}, \qquad x_6^{(1)} = -\frac{9z^3\overline{z}(4+z\overline{z})}{8\sqrt{2}};$$

$$y_1^{(1)} = \frac{9\overline{z}^2\left(-4 + z^2\overline{z}^2\right)}{8\sqrt{2}},$$

$$y_2^{(1)} = -\frac{\overline{z}\left(28 + 18z\overline{z} + 33z^2\overline{z}^2 + 16z^3\overline{z}^3\right)}{8},$$

$$y_3^{(1)} = \frac{9z\bar{z}^2\left(-4+z^2\bar{z}^2\right)}{16}, \qquad y_4^{(1)} = \frac{9\sqrt{31}z^2\bar{z}^2\left(-4+z^2\bar{z}^2\right)}{16\sqrt{14}},$$
$$y_5^{(1)} = \frac{112+56z\bar{z}+96z^2\bar{z}^2-14z^3\bar{z}^3-31z^4\bar{z}^4}{16\sqrt{14}},$$
$$y_6^{(1)} = \frac{z\left(16+36z\bar{z}+78z^2\bar{z}^2+31z^3\bar{z}^3\right)}{16}.$$

Similarly, two local sections $f_2(z,\bar{z})$ and $g_2(z,\bar{z})$ of $\underline{\mathrm{Im}}(\varphi_2)$, which is an anti-holomorphic subbundle of $S^2 \times \mathbf{C}^6$, are given by

$$f_2(z,\bar{z}) = \left(x_1^{(2)},\ x_2^{(2)},\ x_3^{(2)},\ x_4^{(2)},\ x_5^{(2)},\ x_6^{(2)}\right),$$
$$g_2(z,\bar{z}) = \left(y_1^{(2)},\ y_2^{(2)},\ y_3^{(2)},\ y_4^{(2)},\ y_5^{(2)},\ y_6^{(2)}\right),$$

where

$$x_1^{(2)} = 2\bar{z}^2\left(124+128z\bar{z}+31z^2\bar{z}^2\right), \qquad x_2^{(2)} = -72\sqrt{2}z\bar{z}^2,$$
$$x_3^{(2)} = -4\sqrt{2}\bar{z}\left(124+128z\bar{z}+31z^2\bar{z}^2\right), \quad x_4^{(2)} = \frac{4\sqrt{31}\left(28+8z\bar{z}+7z^2\bar{z}^2\right)}{\sqrt{7}},$$
$$x_5^{(2)} = \frac{288z\bar{z}}{\sqrt{7}}, \qquad x_6^{(2)} = -144\sqrt{2}z;$$
$$y_1^{(2)} = -72\sqrt{2}\bar{z}^3, \qquad y_2^{(2)} = 2\bar{z}^2\left(28+248z\bar{z}+31z^2\bar{z}^2\right),$$
$$y_3^{(2)} = 288\bar{z}^2, \qquad y_4^{(2)} = \frac{36\sqrt{62}z\bar{z}^2(8+z\bar{z})}{\sqrt{7}},$$
$$y_5^{(2)} = -\frac{4\sqrt{2}\bar{z}\left(28+248z\bar{z}+31z^2\bar{z}^2\right)}{\sqrt{7}}, \quad y_6^{(2)} = 4\left(28+248z\bar{z}+31z^2\bar{z}^2\right).$$

It is very easy to see that $\mathrm{rank}(\varphi_0) = \mathrm{rank}(\varphi_1) = \mathrm{rank}(\varphi_2) = 2$. Hence $\varphi_0, \varphi_1, \varphi_2 : S^2 \to G_{2,6}$ is a non-degenerate harmonic sequence.

An immediate computation shows that the induced metric by φ_0 is given by

$$ds_0^2 = \frac{4}{(1+z\bar{z})^2}dzd\bar{z},$$

and the induced metric by φ_2 is given by $ds_2^2 = 8\lambda_2^2 dzd\bar{z}$, where $\lambda_2^2 = A/B^2$, $A = 14336 + 65856x + 197904x^2 + 199456x^3 + 90984x^4 + 18228x^5 + 1457x^6$, $B = 112 + 1024x + 1176x^2 + 376x^3 + 31x^4$, $x = z\bar{z}$. By [12] we get that the induced metric ds_1^2 by φ_1 is $ds_0^2 + ds_2^2$. It can easily be checked that curvature $K(\varphi_0)$ of φ_0 is 1, and φ_1 and φ_2 are not maps of constant curvature, namely, $\varphi_0, \varphi_1, \varphi_2 : S^2 \to G_{2,6}$ is not a harmonic sequence of constant curvature.

Furthermore, by a direct computation we get

$$|\varphi_0|^2 = (1+z\overline{z})^4,$$
$$|\varphi_1|^2 = \frac{112+1024z\overline{z}+1176z^2\overline{z}^2+376z^3\overline{z}^3+31z^4\overline{z}^4}{(1+z\overline{z})^4},$$
$$|\varphi_2|^2 = \frac{1}{112+1024z\overline{z}+1176z^2\overline{z}^2+376z^3\overline{z}^3+31z^4\overline{z}^4}.$$

Then by Lemma 3.2 we have

$$|\det\Omega_0|^2dz^2d\overline{z}^2 = \frac{112+1024z\overline{z}+1176z^2\overline{z}^2+376z^3\overline{z}^3+31z^4\overline{z}^4}{(1+z\overline{z})^8}dz^2d\overline{z}^2 \neq 0,$$

$$|\det\Omega_1|^2dz^2d\overline{z}^2 = \frac{(1+z\overline{z})^4}{\left(112+1024z\overline{z}+1176z^2\overline{z}^2+376z^3\overline{z}^3+31z^4\overline{z}^4\right)^2}dz^2d\overline{z}^2 \neq 0.$$

Therefore, this is a totally unramified harmonic sequence.

In [12] a harmonic sequence of constant curvature was given. This shows that the case of complex Grassmann manifolds is very complicated, and it is very difficult for classification of pseudo-holomorphic spheres of constant curvature in a complex Grassmann manifold.

5. Minimal spheres with constant Kähler angle

In this section we will discuss pseudo-holomorphic curves with constant Kähler angle in a complex Grassmann manifold. We know that the pseudo-holomorphic curve of constant curvature has constant Kähler angle (cf. [12]).

Let $\varphi : S^2 \to G_{k,n}$ be a pseudo-holomorphic curve with constant Kähler angle θ, and let $\varphi_0, \varphi_1, \ldots, \varphi_{\alpha_0}$ be its associated harmonic sequence. Let $\varphi = \varphi_\alpha$ for some $\alpha = 0, 1, \ldots, \alpha_0$.

Suppose that $\varphi_0, \varphi_1, \ldots, \varphi_{\alpha_0}$is a totally unramified non-degenerate harmonic sequence, then from results of the above section it follows that

$$t_\alpha = \frac{\alpha(\alpha_0-\alpha+1)}{(\alpha+1)(\alpha_0-\alpha)}. \tag{29}$$

Since t_α is constant, then we have

$$\delta_\alpha l_{\alpha-1} = \delta_{\alpha-1} l_\alpha.$$

Then by (16) it is not difficult to get that there exists a non-zero constant c such that

$$\left|F^{(\alpha)}\right|^{2\delta_{\alpha-1}} = c\left|F^{(\alpha-1)}\right|^{2\delta_\alpha}. \tag{30}$$

Thus we can prove the following theorem, which is similar to Theorem 9.2 in [2].

THEOREM 5.1: *Let $\varphi : S^2 \to G_{k,n}$ be a linearly full pseudo-holomorphic curve with constant Kähler angle, and let $\varphi_0, \varphi_1, \dots, \varphi_{\alpha_0}$ be its associated harmonic sequence with $\varphi = \varphi_\alpha$. If δ_α and $\delta_{\alpha-1}$ are coprime, then φ is of constant curvature.*

Proof. By (30), $\left|F^{(\alpha)}\right|^2$ and $\left|F^{(\alpha-1)}\right|^2$, as elements of $\mathbf{C}[z,\overline{z}]$, have the same prime factors. Suppose that $P(z,\overline{z})$ is a prime factor with degree d in z. Then we have

$$\delta_{\alpha-1}|d, \quad \delta_\alpha|d.$$

Since δ_α and $\delta_{\alpha-1}$ are coprime, then d must be 1. Therefore, we have

$$\left|F^{(\alpha-1)}\right|^2 = P(z,\overline{z})^{\delta_{\alpha-1}}, \quad \left|F^{(\alpha)}\right|^2 = P(z,\overline{z})^{\delta_\alpha}.$$

Without loss of generality, we may assume that $P(z,\overline{z}) = A + \overline{B}z + B\overline{z} + Dz\overline{z}$ for some complex numbers A, B, D with A, D real, and $AD - B\overline{B} > 0$. Then, using (16) we have

$$l_{\alpha-1} + l_\alpha = (\delta_{\alpha-1} + \delta_\alpha)\partial\overline{\partial}\log P(z,\overline{z}) = (\delta_{\alpha-1} + \delta_\alpha)\frac{AD - B\overline{B}}{P(z,\overline{z})^2},$$

which implies that $ds_\alpha^2 = (l_{\alpha-1} + l_\alpha)dzd\overline{z}$ is of constant curvature. ∎

In fact, if $\varphi_0, \varphi_1, \dots, \varphi_{\alpha_0}$ is a totally unramified non-degenerate harmonic sequence, then from Theorem 4.1 it follows that

$$(\delta_{\alpha-1}, \delta_\alpha) = k\left(\alpha(\alpha_0 - \alpha + 1), (\alpha+1)(\alpha_0 - \alpha)\right) \geq k.$$

Therefore, it is impossible that $\delta_{\alpha-1}$ and δ_α are coprime for totally unramified non-degenerate harmonic sequence $\varphi_0, \varphi_1, \dots, \varphi_{\alpha_0} : S^2 \to G_{k,n}$ $(2 \leq k \leq n-2)$.

If α_0 and $\alpha_0 + 2$ are consecutive prime integers, then

$$(\delta_{\alpha-1}, \delta_\alpha) = k(\alpha(\alpha+1), \alpha_0 - 2\alpha) = k(\alpha+1, \alpha_0+2) = k.$$

Hence, when k is a prime number, above d must 1 or k, namely, $\left|F^{(\alpha)}\right|^2$ and $\left|F^{(\alpha-1)}\right|^2$ possibly have prime factors with degree k in z. Thus we get the following theorem.

THEOREM 5.2: *Let $\varphi : S^2 \to G_{k,n}$ be a linearly full pseudo-holomorphic curve of constant Kähler angle with a non-degenerate associated harmonic sequence*

$\varphi_0, \varphi_1, \ldots, \varphi_{\alpha_0}$. *Suppose that* $\varphi_0, \varphi_1, \ldots, \varphi_{\alpha_0}$ *is totally unramfied with* $\varphi = \varphi_\alpha$ *for some* $\alpha = 0, 1, \ldots, \alpha_0 - 1$. *If* α_0 *and* $\alpha_0 + 2$ *are consecutive prime integers, and if* k *is a prime number, then* φ *is of constant curvature unless* $|\varphi|^2$ *or* $|\varphi|^{-2}$ *has prime factors with degree* k $(2 \leq k \leq n-2)$ *in* z.

Theorem 5.1 and 5.2 are generalizations of Bolton's results (cf. [2]).

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